Topology and its Applications 159 (2012) 2757–2763

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Browder's convergence theorem for multivalued mappings without endpoint condition

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ARTICLE INFO

Article history:
Received 6 February 2012
Accepted 8 March 2012

Dedicated to W.A. Kirk on the occasion of his 75th birthday

MSC:
47H09
47H10

Keywords:
Fixed point
Multivalued nonexpansive mapping
Strong convergence
$\mathbb{R}$-tree

ABSTRACT

We prove Browder's convergence theorem for multivalued nonexpansive mappings in a complete $\mathbb{R}$-tree without endpoint condition. This gives an affirmative answer to Jung's question for nonlinear spaces.

1. Introduction

Let E be a nonempty subset of a geodesic metric space (X,d). We shall denote the family of nonempty bounded closed subsets of E by $BC(E)$, the family of nonempty bounded closed convex subsets of E by $BCC(E)$, the family of nonempty compact subsets of E by $K(E)$ and the family of nonempty compact convex subsets of E by $KC(E)$. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $BC(X)$, i.e.,

$$H(A,B)=\max\Big\{\sup_{a\in A}\operatorname{dist}(a,B),\sup_{b\in B}\operatorname{dist}(b,A)\Big\},\quad A,B\in BC(X).$$

A multivalued mapping $T:E\to BC(X)$ is said to be a *contraction* if there exists a constant $k\in[0,1)$ such that

$$H\big(T(x),T(y)\big)\leqslant kd(x,y),\quad x,\ y\in E. \tag{1.1}$$

If (1.1) is valid when $k=1$, then T is called *nonexpansive*. A point $x\in E$ is called a *fixed point* of T if $x\in T(x)$. A point $x\in E$ is said to be an *endpoint* of T if x is a fixed point of T and $T(x)=\{x\}$ (see [29]). We shall denote by $Fix(T)$ the set of all fixed points of T and by $End(T)$ the set of all endpoints of T. We see that for each mapping T, $End(T)\subseteq Fix(T)$ and the converse is not true in general. A mapping T is said to satisfies the *endpoint condition* if $End(T)=Fix(T)$.

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2012.03.006

Let E be a nonempty closed convex subset of a Banach space X and let $T : E \to E$ be a single-valued nonexpansive mapping. Fix $u \in E$, for each $s \in (0,1)$, we can define a contraction $t_s : E \to E$ by

$$t_s(x) = su + (1-s)t(x), \quad x \in E.$$

Then by Banach's contraction principle, t_s has a unique fixed point $x_s \in E$, that is,

$$x_s = su + (1-s)t(x_s). \tag{1.2}$$

In 1967, Browder [5] proved the following theorem.

Theorem 1.1. *Let E be a nonempty bounded closed convex subset of a Hilbert space H and $t : E \to E$ be nonexpansive. Fix $u \in E$ and let $\{x_s\}$ be defined by (1.2). Then $\{x_s\}$ converges strongly as $s \to 0$ to the point of $\mathrm{Fix}(t)$ nearest to u.*

Let $T : E \to BC(E)$ be a multivalued nonexpansive mapping. Fix $u \in E$, for each $s \in (0,1)$, we define a contraction $G_s : E \to BC(E)$ by

$$G_s(x) = su + (1-s)T(x), \quad x \in E. \tag{1.3}$$

Then by Nadler's theorem [20], G_s has a (not necessary unique) fixed point $x_s \in E$, that is,

$$x_s \in su + (1-s)T(x_s). \tag{1.4}$$

A natural question arises whether Browder's theorem can be extended to the multivalued case. The first result concerning to this question was proved by Lopez and Xu [17] in 1995. They gave the strong convergence of the net $\{x_s\}$ defined by (1.4) under the endpoint condition. Since then the strong convergence of $\{x_s\}$ has been developed and many of papers have appeared (see e.g., [14,24,13,26,27]). Among other things, Jung [13] obtained the following result.

Theorem 1.2. *Let X be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, E a nonempty closed convex subset of X, and $T : E \to K(E)$ a nonexpansive mapping. Suppose that T satisfies the endpoint condition. Fix $u \in E$ and let $\{x_s\}$ be defined by (1.4). Then T has a fixed point if and only if $\{x_s\}$ remains bounded as $s \to 0$ and in this case, $\{x_s\}$ converges strongly as $s \to 0$ to a fixed point of T.*

Jung also posed an open question whether the endpoint condition in Theorem 1.2 can be omitted. In view of Pietramala's example [22], Shahzad and Zegeye [26] pointed out that it is almost impossible to completely omit this condition for nonexpansive multivalued mappings even in the Euclidean plane $\mathbb{R}^2$. They also improved Jung's theorem under some mild conditions. On the other hand, the present authors [6] extended Jung's theorem to a special kind of metric spaces, namely, CAT(0) spaces.

Theorem 1.3. *Let E be a nonempty closed convex subset of a complete CAT(0) space X and $T : E \to K(E)$ be a nonexpansive mapping. Suppose that T satisfies the endpoint condition. Fix $u \in E$ and let $\{x_s\}$ be defined by (1.4). Then T has a fixed point if and only if $\{x_s\}$ remains bounded as $s \to 0$. In this case, the following statements hold:*

(i) *$\{x_s\}$ converges to the unique fixed point z of T which is nearest u.*
(ii) *If $\{u_n\}$ is a bounded sequence in C having $\lim_{n\to\infty} \mathrm{dist}(u_n, T(u_n)) = 0$, then*

$$d^2(u,z) \leqslant \mu_n d^2(u,u_n)$$

for all Banach limits μ.

It is well known that the class of Hilbert spaces is a subclass of CAT(0) spaces (see [4]). Thus, we cannot omit the endpoint condition in Theorem 1.3. Summary: there is no any result concerning Browder's convergence theorem in linear or nonlinear spaces which completely removes the endpoint condition. However, there is a nice subclass of CAT(0) spaces, namely $\mathbb{R}$-trees, such that Browder's theorem holds without this condition.

2. Preliminaries

For any pair of points x, y in a metric space (X,d), a *geodesic path* joining these points is an isometry c from a closed interval $[0,l]$ to X such that $c(0) = x$ and $c(l) = y$. The image of c is called a *geodesic segment* joining x and y. If there exists exactly one geodesic joining x and y we denote by $[x,y]$ the geodesic joining x and y. For $x, y \in X$ and $\alpha \in [0,1]$, we denote the point $z \in [x,y]$ such that $d(x,z) = \alpha d(x,y)$ by $(1-\alpha)x \oplus \alpha y$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining

x and y for each $x, y \in X$. A subset E of X is said to be *convex* if E includes every geodesic segment joining any two of its points, and E is said to be *gated* if for any point $x \notin E$ there is a unique point y_x such that for any $z \in E$,

$$d(x,z) = d(x, y_x) + d(y_x, z).$$

The point y_x is called the gate of x in E. From the definition of y_x we see that it is also the unique nearest point of x in E.

$\mathbb{R}$-trees (sometimes called metric trees) were introduced by Tits [28] in 1977. Fixed point theory in $\mathbb{R}$-trees was first studied by Kirk [15]. He showed that every continuous mapping defined on a geodesically bounded complete $\mathbb{R}$-tree always has a fixed point. Since then fixed point theorems for various types of mappings in $\mathbb{R}$-trees has been developed (see e.g., [10,1,19,21,25,3,2]).

Definition 2.1. An $\mathbb{R}$-tree is a geodesic metric space X such that:

(i) there is a unique geodesic segment $[x, y]$ joining each pair of points $x, y \in X$;
(ii) if $[y,x] \cap [x,z] = \{x\}$, then $[y,x] \cup [x,z] = [y,z]$.

By (i) and (ii) we have

(iii) if $u, v, w \in X$, then $[u,v] \cap [u,w] = [u,z]$ for some $z \in X$.

An $\mathbb{R}$-tree is a special case of a CAT(0) space. For a thorough discussion of these spaces and their applications, see [4]. It is known that in an $\mathbb{R}$-tree the gated subsets are precisely the closed convex subsets (see [10]). We now collect some basic properties of $\mathbb{R}$-trees.

Lemma 2.2. *Let X be a complete $\mathbb{R}$-tree. Then the following statements hold*:

(i) *[9, Lemma 2.5] if $x, y, z \in X$ and $\alpha \in [0,1]$, then*
$$d^2\big(\alpha x \oplus (1-\alpha)y, z\big) \leqslant \alpha d^2(x,z) + (1-\alpha)d^2(y,z) - \alpha(1-\alpha)d^2(x,y);$$
(ii) *[9, Lemma 2.3] if $x, y, z \in X$, then $d(x,z) + d(z,y) = d(x,y)$ if and only if $z \in [x,y]$;*
(iii) *[1, Lemma 2.1] if $x, y \in X$ and $z \in [x,y]$, then $[x,z] \subseteq [x,y]$;*
(iv) *[19, Lemma 3.1] if A and B are bounded closed convex subsets of X, then, for any $u \in X$, $d(x,y) \leqslant H(A,B)$, where the points x, y are respectively the unique nearest points of u in A and B;*
(v) *[18, Proposition 1] if E is a nonempty closed convex subset of X and $T : E \to BCC(E)$ is a nonexpansive mapping, then Fix(T) is closed and convex.*

Let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r\big(x, \{x_n\}\big) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\big(\{x_n\}\big) = \inf\big\{r\big(x, \{x_n\}\big)\colon x \in X\big\},$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\big(\{x_n\}\big) = \big\{x \in X\colon r\big(x, \{x_n\}\big) = r\big(\{x_n\}\big)\big\}.$$

It is known from Proposition 7 of [8] that in an $\mathbb{R}$-tree, $A(\{x_n\})$ consists of exactly one point. The following lemma can be found in [7].

Lemma 2.3. *([7, Proposition 2.1]) If E is a closed convex subset of X and if $\{x_n\}$ is a bounded sequence in E, then $A(\{x_n\})$ is in E.*

Recall that a bounded sequence $\{x_n\}$ in X is said to be *regular* if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$. Every bounded sequence in a complete $\mathbb{R}$-tree has a regular subsequence (see e.g., [16,11]).

The following lemmas are also needed.

Lemma 2.4. *([6, Lemma 3.2]) Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X and $T : E \to K(E)$ be a nonexpansive mapping. Suppose $\{x_n\}$ is a sequence in E which is regular and $\lim_{n\to\infty} \operatorname{dist}(x_n, T(x_n)) = 0$. If $A(\{x_n\}) = \{z\}$, then z is a fixed point of T.*

Lemma 2.5. *([23, Lemma 2.1]) Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X and $t : E \to E$ be a nonexpansive mapping. Let $u \in E$ be fixed. For each $s \in (0, 1)$, the mapping $t_s : E \to E$ defined by*

$$t_s(x) = su \oplus (1 - s)t(x) \quad \text{for } x \in E$$

has a unique fixed point.

Recall that a continuous linear functional μ on ℓ_∞, the Banach space of bounded real sequences, is called a *Banach limit* if $\|\mu\| = \mu(1, 1, \ldots) = 1$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for all $\{a_n\} \in \ell_\infty$.

Lemma 2.6. *([23, Lemma 2.2]) Let E, X, t be as in Lemma 2.5. For each $s \in (0, 1)$, let x_s be the fixed point of t_s, that is,*

$$x_s = t_s(x_s) = su \oplus (1 - s)t(x_s). \tag{2.1}$$

Then Fix$(t) \neq \emptyset$ if and only if $\{x_s\}$ given by the formula (2.1) remains bounded as $s \to 0$. In this case, the following statements hold:

(i) *$\{x_s\}$ converges to the unique fixed point z of t which is nearest u;*
(ii) *$d^2(u, z) \leqslant \mu_n d^2(u, u_n)$ for all Banach limits μ and all bounded sequences $\{u_n\}$ with $\lim_{n\to\infty} d(u_n, t(u_n)) = 0$.*

Lemma 2.7. *([23, Theorem 2.3]) Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X and $t : E \to E$ be a nonexpansive mapping for which Fix$(t) \neq \emptyset$. Suppose that $u, z_1 \in E$ are arbitrarily chosen and $\{z_n\}$ is defined by*

$$z_{n+1} = \alpha_n u \oplus (1 - \alpha_n)t(z_n), \quad n \geqslant 1, \tag{2.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$.

Then $\{z_n\}$ converges to the unique point of Fix(t) which is nearest to u.

3. Main results

Let E be a nonempty closed convex subset of an $\mathbb{R}$-tree X. For $x \in X$, we denote by $P_E(x)$ the unique nearest point of x in E. The following lemma can be found in [1].

Lemma 3.1. *Let E be a nonempty closed convex subset of an $\mathbb{R}$-tree X. Then, for any $x, y \in X$, we have either*

$$P_E(x) = P_E(y)$$

or

$$d(x, y) = d\big(x, P_E(x)\big) + d\big(P_E(x), P_E(y)\big) + d\big(P_E(y), y\big).$$

As a consequence of Lemma 3.1, we obtain

Lemma 3.2. *Let E be a nonempty subset of a complete $\mathbb{R}$-tree X and $T : E \to BCC(E)$ be a nonexpansive mapping such that Fix$(T) \neq \emptyset$. Let $u \in E$ and $z \in$ Fix(T). Then for each $x \in E$, we have either*

$$d\big(z, P_{T(x)}(u)\big) \leqslant d(z, x) \quad \text{or } P_{T(x)}(u) \in [u, z].$$

Proof. By Lemma 3.1, we have either

$$P_{T(x)}(u) = P_{T(x)}(z)$$

or

$$d(u, z) = d\big(u, P_{T(x)}(u)\big) + d\big(P_{T(x)}(u), P_{T(x)}(z)\big) + d\big(P_{T(x)}(z), z\big).$$

If $P_{T(x)}(u) = P_{T(x)}(z)$, then

$$d\big(z, P_{T(x)}(u)\big) = \operatorname{dist}\big(z, T(x)\big) \leqslant H\big(T(z), T(x)\big) \leqslant d(z, x).$$

If $P_{T(x)}(u) \neq P_{T(x)}(z)$, then

$$d(u,z) = d\big(u, P_{T(x)}(u)\big) + d\big(P_{T(x)}(u), P_{T(x)}(z)\big) + d\big(P_{T(x)}(z), z\big)$$
$$= d\big(u, P_{T(x)}(u)\big) + d\big(P_{T(x)}(u), z\big).$$

By Lemma 2.2(ii), we have $P_{T(x)}(u) \in [u,z]$. □

Lemma 3.3. *Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X and $T : E \to BCC(E)$ be a multivalued nonexpansive mapping. Fix $u \in E$ and define $f : E \to E$ by $f(x) = P_{T(x)}(u)$ for $x \in E$. For each $s \in (0,1)$, we define $t_s : E \to E$ by*

$$t_s(x) = su \oplus (1-s)f(x), \quad x \in E.$$

Then t_s has a unique fixed point.

Proof. By Lemma 2.2(iv), f is nonexpansive. The conclusion follows from Lemma 2.5. □

Lemma 3.4. *Let E, X, T, u, f be as in Lemma 3.3. For each $s \in (0,1)$, let x_s be the fixed point of t_s, that is,*

$$x_s = su \oplus (1-s)f(x_s). \tag{3.1}$$

If $z \in Fix(T)$, then $x_s \in [u,z]$, equivalently,

$$d(u,x_s) + d(x_s,z) = d(u,z). \tag{3.2}$$

Proof. If $u \in Fix(T)$, then $u = P_{T(u)}(u) = f(u)$. Thus

$$d\big(f(x_s),u\big) = d\big(f(x_s), f(u)\big) \leqslant H\big(T(x_s), T(u)\big) \leqslant d(x_s,u) = sd\big(f(x_s),u\big).$$

This implies that $u = f(x_s) = x_s$. Then the conclusion follows. Now, if $u \notin Fix(T)$, then $x_s \neq f(x_s)$. Otherwise, $d(u, f(x_s)) = d(u,x_s) = (1-s)d(u, f(x_s))$ which implies $u = f(x_s) = x_s$, contradicting the fact that $u \notin Fix(T)$. By Lemma 3.2, we have either

$$d\big(z, f(x_s)\big) \leqslant d(z,x_s) \quad \text{or } f(x_s) \in [u,z].$$

If $d(z, f(x_s)) \leqslant d(z,x_s)$, then the gate of z in $[u, f(x_s)]$ lies in $(x_s, f(x_s)]$. Hence, in any case, $x_s \in [u,z]$. □

Lemma 3.5. *Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X and $T : E \to KC(E)$ be a nonexpansive mapping. Fix $u \in E$ and define $f : E \to E$ by $f(x) = P_{T(x)}(u)$. Then the following statements hold:*

(i) *$Fix(f) \neq \emptyset$ if and only if $Fix(T) \neq \emptyset$;*
(ii) *if $Fix(f) \neq \emptyset$ and x and y are respectively the unique nearest points of u in $Fix(f)$ and $Fix(T)$, then $x = y$.*

Proof. (i) Clearly $Fix(f) \subseteq Fix(T)$. Thus one direction is obvious. Conversely, let $p \in Fix(T)$. Then by (3.2), $d(x_s,p) \leqslant d(u,p)$ for all $s \in (0,1)$. This implies that $\{x_s\}$ is bounded. Hence by Lemma 2.6, $Fix(f) \neq \emptyset$.

(ii) Let a sequence $\{s_n\}$ in $(0,1)$ converges to 0 and $x_n := x_{s_n}$. Then obtain a regular subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and denote $A(\{x_{n_k}\}) = \{v\}$. For $k \in \mathbb{N}$, $x_{n_k} = s_{n_k}u \oplus (1-s_{n_k})f(x_{n_k})$. Thus

$$d\big(x_{n_k}, f(x_{n_k})\big) = s_{n_k} d\big(u, f(x_{n_k})\big) \to 0 \quad \text{as } k \to \infty.$$

By Lemma 2.4, $v \in Fix(f) \subseteq Fix(T)$. Next, we show that $\{x_{n_k}\}$ converges to v. Suppose that $\varepsilon = \limsup_{k\to\infty} d^2(x_{n_k}, v) > 0$ and denote $r = \limsup_{k\to\infty} d^2(x_{n_k}, u)$. Then $0 < \varepsilon < r$. Let $\alpha \in (0,1)$ be such that $0 < \alpha r < \varepsilon$. This implies that

$$\alpha^2 r < \alpha\varepsilon. \tag{3.3}$$

Let $w = \alpha u \oplus (1-\alpha)v$. Then by Lemma 2.2(i) and (3.2), we have

$$d^2(x_{n_k}, w) \leqslant \alpha d^2(x_{n_k}, u) + (1-\alpha)d^2(x_{n_k}, v) - \alpha(1-\alpha)d^2(u,v)$$
$$< \alpha^2 d^2(x_{n_k}, u) + (1-\alpha)^2 d^2(x_{n_k}, v).$$

This together with (3.3) imply that

$$\limsup_{n\to\infty} d^2(x_{n_k}, w) \leqslant \alpha^2 r + (1-\alpha)\varepsilon < \varepsilon,$$

contradicting the fact that $A(\{x_{n_k}\}) = \{v\}$. Therefore $\lim_{k\to\infty} x_{n_k} = v$. Since $x, y \in Fix(T)$, then by (3.2), we have

$$d(u, x_{n_k}) + d(x_{n_k}, x) = d(u, x)$$

and

$$d(u, x_{n_k}) + d(x_{n_k}, y) = d(u, y).$$

These imply, by letting $k \to \infty$, that

$$d(u, v) + d(v, x) = d(u, x) \tag{3.4}$$

and

$$d(u, v) + d(v, y) = d(u, y). \tag{3.5}$$

Since x and y are respectively the unique nearest points of u in $Fix(f)$ and $Fix(T)$, by (3.4) and (3.5) we have $x = v = y$ and the proof is complete. □

The following theorem is our main result.

Theorem 3.6. *Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X, $T : E \to KC(E)$ be a multivalued nonexpansive mapping and $u \in E$. Then T has a fixed point if and only if $\{x_s\}$ given by* (3.1) *remains bounded as $s \to 0$. In this case, the following statements hold*:

(i) *$\{x_s\}$ converges to the unique fixed point z of T which is nearest u.*
(ii) *If $\{u_n\}$ is a bounded sequence in E having $\lim_{n\to\infty} \operatorname{dist}(u_n, T(u_n)) = 0$, then*

$$d^2(u, z) \leqslant \mu_n d^2(u, u_n)$$

for all Banach limits μ.

Proof. We note that $x_s = su \oplus (1 - s) f(x_s)$ and f is nonexpansive. Thus by Lemma 2.6 and Lemma 3.5(i), T has a fixed point if and only if $\{x_s\}$ is bounded.

(i) Follows from Lemma 2.6(i) and Lemma 3.5(ii).

(ii) The proof is similar to the one given in [6]. For convenience of the reader we include the details. Let $\{u_n\}$ be a bounded sequence in E such that $\lim_n \operatorname{dist}(u_n, T(u_n)) = 0$. Let μ be a Banach limit and suppose $\mu_n d^2(u, u_n) < \rho < \gamma < d^2(u, z)$. Thus there exists a subsequence $\{u_{n_k}\}$ with

$$d^2(u, u_{n_k}) < \gamma \quad \text{for all } k. \tag{3.6}$$

Otherwise $d^2(u, u_n) \geqslant \gamma$ for all large n which implies $\mu_n d^2(u, u_n) \geqslant \gamma > \rho$, a contradiction, and therefore (3.6) holds. We can assume that $\{u_{n_k}\}$ is a regular subsequence. Since $\lim_{k\to\infty} \operatorname{dist}(u_{n_k}, T(u_{n_k})) = 0$, if $A(\{u_{n_k}\}) = \{w\}$, then $w \in Fix(T)$ by Lemma 2.4. Then by (3.6) and Lemma 2.3, $w \in \overline{B}(u, \sqrt{\gamma})$ which is contradicting to the fact that z is the nearest point in $Fix(T)$ to u. This concludes that $d^2(u, z) \leqslant \mu_n d^2(u, u_n)$. □

Let $T : E \to KC(E)$ be a nonexpansive mapping and $\{\alpha_n\}$ be a sequence in $(0, 1)$. Fix $u, z_1 \in C$. Let $y_1 \in T(z_1)$ be the gate of u in $T(z_1)$. Define

$$z_2 = \alpha_1 u \oplus (1 - \alpha_1) y_1.$$

Let $y_2 \in T(z_2)$ be the gate of u in $T(z_2)$. Inductively, we have

$$z_{n+1} = \alpha_n u \oplus (1 - \alpha_n) y_n, \tag{3.7}$$

where $y_n \in T(z_n)$ is the gate of u in $T(z_n)$.

Now, we obtain a strong convergence theorem of Halpern's iteration [12] for multivalued nonexpansive mappings.

Theorem 3.7. *Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X, $T : E \to KC(E)$ be multivalued nonexpansive mapping such that $Fix(T) \neq \emptyset$. Suppose that $u, z_1 \in E$ are arbitrarily chosen and $\{z_n\}$ is defined by* (3.7)*, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$.

Then $\{z_n\}$ converges to the unique point of $Fix(T)$ which is nearest to u.

Proof. We observe that $z_{n+1} = \alpha_n u \oplus (1-\alpha_n) f(z_n)$. The conclusion follows from Lemmas 2.7 and 3.5. $\square$

Finally, we finish the paper with the following question:

Question 3.8. In the original Jung's theorem, the mapping T is assumed to take compact values while in Theorem 3.6 T takes compact and convex values. Does Theorem 3.6 hold if T is only assumed to take compact values?

Acknowledgement

We thank the National Research University Project under Thailand's Office of the Higher Education Commission for financial support.

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